

Anderson Localization and Beyond

Ahmed Akhtar

December 2, 2018

Contents

1	Introduction	1
2	Foundational work on transport in disordered lattices	2
3	Further theoretical work	6
4	Experimental confirmations of localization	7

1 Introduction

Setting the agenda for many active areas of research in condensed matter physics, Philip Warren Anderson is arguably one of the most influential physicists of our time. He has completed groundbreaking, foundational work in disordered systems, beginning with single-particle localization (the subject of this paper); particle physics, through his introduction of spontaneous symmetry breaking; and superconductivity. He did much of this work at the famed Bell Labs institute, from 1949 to 1984. He was a professor of theoretical physics at Cambridge from 1967 to 1975 before moving to Princeton, where he is currently the Joseph Henry Professor of Physics, Emeritus.

In 1958, Anderson pioneered the first substantial work on transport in disordered lattices. He initially was trying to understand Feher's experiments, which were also done at Bell labs, that indicated abnormally slow electron spin relaxation times in phosphorous doped silicon semi-conductors [1]. Semi-conductor doping is the intentional addition of impurities into a semi-conductor to change its electrical properties. In particular, doping a semi-conductor can increase its conductivity by introducing more carriers into the metal, called donors. The donors' hyperfine interaction with the surrounding nuclei resembles something like a random field and the energy of the carrier at that site can therefore be viewed as a stochastic variable. Recognizing the significance of the randomness, Anderson set out to develop a theory for describing transport in such a disordered system. Interestingly, Anderson found that, for a large class of systems, there was a critical disorder strength at which transport ceases to occur [2].

2 Foundational work on transport in disordered lattices

The stated purpose of [2] was to provide a foundation for the theory of transport in disordered lattices, and so Anderson considers a simple, ubiquitous model: a lattice where each site can be occupied by a single spin with an on-site random disorder potential, and each pair of sites has an isotropic interaction potential. Physically, the sites could represent impurity sites, the random energies could be the hyperfine interactions with surrounding nuclei, so that, for example, a spin occupying site j has a random energy E_j , and there is an interaction $V_{jk}(r_{jk})$ between sites that allows spins to “move” through the lattice by means of swapping opposite spins on neighboring sites. Note that there is no external reservoir or heat bath in this problem—we will find the system fails to act as one for any subsystem. Indeed, for low enough densities and short-ranged interactions, there is no transport.

The Schrodinger equation for this model is

$$i\dot{a}_j = E_j a_j + \sum_{k \neq j} V_{jk} a_k \quad (1)$$

We are interested in the long time behavior of a_j when the particle begins localized at $j = 0$ i.e. $a_0(0) = 1, a_j(0) = 0, j \neq 0$. Accordingly, it will be easier to consider the corresponding Laplace variables

$$f_j(s) = \int_0^\infty e^{-st} a_j(t) dt \quad (2)$$

because of the following observation:

$$\lim_{s \rightarrow 0^+} s f_j(s) = \lim_{s \rightarrow 0^+} \int_0^\infty s e^{-st} a_j(t) dt = \lim_{s \rightarrow 0^+} -e^{-st} a_j(t) \Big|_0^\infty + \int_0^\infty e^{-st} \dot{a}_j(t) dt = a_j(\infty) \quad (3)$$

Thus, studying the small s behavior of the conjugate variable $f_j(s)$ reveals the large t behavior of $a_j(t)$. Thus we begin by taking the Laplace transform of the Schrodinger equation. We can then plug in the initial values and get a relation between the different conjugate variables.

$$i(s f_j(s) - a_j(0)) = E_j f_j + \sum_{k \neq j} V_{jk} f_k \quad (4)$$

$$f_j(s) = \frac{i\delta_{0j}}{is - E_j} + \sum_{k \neq j} \frac{V_{jk}}{is - E_j} f_k(s) \quad (5)$$

We can then solve the equation for $f_0(s)$ by repeatedly plugging in the above equation, getting a series $V_c(0)$ summing over closed paths at 0 weighted at each edge by the interaction coefficients V_{jk} .

$$f_0(s) = \frac{i}{is - E_0} + \sum_{k \neq 0} \frac{V_{0k}}{is - E_0} \sum_{l \neq k} \frac{V_{kl}}{is - E_k} \left(\frac{i\delta_{0l}}{is - E_l} + \sum_{m \neq l} \frac{V_{lm}}{is - E_l} (\dots) \right) \quad (6)$$

$$= \frac{i}{is - E_0} \left(1 - i \left(\sum_k \frac{V_{0k}^2}{is - E_k} + \sum_{k,l} \frac{V_{0k}V_{kl}V_{l0}}{(is - E_k)(is - E_l)} + \dots \right) f_0(s) \right) \quad (7)$$

$$\equiv \frac{i}{is - E_0} (1 - iV_c(0)f_0(s)) \quad (8)$$

$$\boxed{f_0(s) = \frac{1}{s + iV_c(0) + iE_0}} \quad (9)$$

The series $V_c(0)$ represents the interference between the wave-function and itself. In the limit $s \rightarrow 0$, we are interested in the lowest order terms. We can rewrite it in terms of its real and imaginary parts. Unsurprisingly, since $V_c(0)$ is a series in the interaction strength, the real part gives the second order perturbation to the random disorder potential $\Delta E^{(2)}$ as $s \rightarrow 0$. The imaginary part carries an extra factor of s , and will therefore be subdominant unless the real part vanishes, in which case we must include it. It can be rewritten in terms of the usual transition probability formula using a property of the Lorentzian distribution, and an additional term for sites with nonzero random potential.

$$V_c(0) \sim \sum_k V_{0k}^2 \left(\frac{-E_k}{s^2 + E_k^2} - i \frac{s}{s^2 + E_k^2} \right) = -\Delta E^{(2)} - i \sum_k V_{0k}^2 \delta(E_k) - is \sum_{k, E_k \neq 0} \frac{V_{0k}^2}{E_k^2} \quad (10)$$

$$V_c(0) \sim -\Delta E^{(2)} - \frac{i}{\tau} - isK \quad (11)$$

The last relation defines K and τ . Of course, since we are only interested in the leading behavior, if τ is finite, we can drop the K term, and if τ is infinite, we can keep the K term.

$$f_0(s) = \frac{i}{is(1 + K) + (i/\tau) - (E_0 - \Delta E^{(2)})} \quad (12)$$

In order to understand the behavior at large t , it is necessary to understand V_c as a probability variable. The imaginary part of V_c is the most relevant since the real part just shifts the mean of the random energies. We will define $X(s)$, a quantity which describes the ‘‘spreading’’ of the wave-function at infinity¹. *A finite value of $X(s)$ as $s \rightarrow 0$ suggests that no real transport occurs.*

$$\text{Im}\{V_c\} = -s \sum_k \frac{|V_{0k}|^2}{s^2 + E_k^2} \equiv -sX(s) = -s \left(\sum_{j \neq 0} \frac{|f_j(s)|^2}{|f_0(s)|^2} + \mathcal{O}(s^2) \right) \quad (13)$$

Thus, the theorem at hand hinges on the convergence of $X(s)$. In evaluating $X(s)$, we will assume a uniform number density n .

¹This relationship can be seen by rewriting it using a simple relation derived from the Laplace-transformed equation of motion.

$$X(s) = \sum_k \frac{V_{0k}^2}{s^2 + E_k^2} = n \int d^3r \frac{V(r)^2}{s^2 + E(r)^2} \quad (14)$$

$X(s)$ is a probability variable in the distribution of random on-site potentials. It will be helpful to label it with the letter E , to indicate a particular value of $X(s)$ for a particular distribution of $\{E_j\}$. It will be helpful to know $P(X)$, the probability that $X(s)$ takes some value X . Towards this end, we will also define an integration measure $d\Omega_E$ for the probability distribution on $\{E_j\}$.

$$P(X) = \int d\Omega_E \delta(X - X_E(s)) = \int d\Omega_E \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ix(X - X_E(s))} \quad (15)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ixX} \int d\Omega_E e^{-ixn \int d^3r \frac{V(r)^2}{s^2 + E(r)^2}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ixX} \langle e^{-ixn \int d^3r \frac{V(r)^2}{s^2 + E(r)^2}} \rangle \quad (16)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ixX} e^{-n \langle \int \left(1 - \exp\left(-ix \frac{V(r)^2}{s^2 + E(r)^2}\right)\right) d^3r \rangle} \quad (17)$$

In the last step we used the Holtsmark-Markoff method result. The integral I in the exponent above can be understood asymptotically using the stationary phase approximation. Considering the case where $s = 0$ first, and expanding around $x = 0$, we see that only small values of E are important, and the variation of $P(E)$ can be neglected, replacing it by $1/W$ where W is the width of the distribution $P(E)$.

$$I = \int d\Omega_E \int dr 4\pi r^2 (1 - \exp(-ixV(r)^2/(E^2 + s^2))) \quad (18)$$

$$\sim \frac{4\pi}{W} \int_0^{\infty} r^2 dr \int_{-\infty}^{\infty} dE (1 - \exp(-ixV(r)^2/(E^2))) \quad (19)$$

$$= 2 \left(\frac{x}{i}\right)^{1/2} \Gamma(1/2) \frac{\langle V(r)^2 \rangle}{W} \quad (20)$$

The convergence depends on how $V(r)$ depends on r . For $V(r) \propto r^{-(3+\varepsilon)}$, for large values of X , $P(X)$ falls off as $X^{-3/2}$, with a finite most probable value (though the mean is divergent). For $V(r) = Ar^{-3}$, we see a logarithmic singularity in the most probable value as $s \rightarrow 0$. This is harder to evaluate. For a boxed distribution of width W , i.e. $E \in [-W/2, W/2]$ and $P(E) = \frac{1}{W}$, we find the most probable value of X goes as

$$[X(s)]_{M.P.} \sim \left(\frac{nA}{W} \sinh^{-1}(W/2s)\right)^2 \quad (21)$$

Although for $V(r) \in \mathcal{O}(r^{3+\varepsilon})$ we found a finite most probable value for $X(s)$ to first order in s , the full series might still diverge, and so it will be necessary to evaluate the sum. [2] goes through significant effort to remove terms in the sum that cause it to blow up. This will change the sum slightly, so that no single term contains any repeating indices.

$$V_c(0) = \sum_{j,k,l,\dots,m \neq 0} V_{0j} \frac{1}{is - E_j} V_{jk} \frac{1}{is - E_k} V_{kl} \frac{1}{is - E_l} \dots \frac{1}{is - E_m} V_{m0} \frac{1}{is - E_0} \quad (22)$$

We first focus on products of L terms which corresponds to paths of length L through the lattice. How many of the products of L terms take a value between T and $T + dT$? Let $n(T)dT$ denote the answer to the previous question. Anderson is able to find $n(T)$ explicitly in some cases.

In the first case, suppose we use a box distribution of width W like the one described earlier and that the potential is nearest neighbor, with a constant value V between neighbors. Then each term takes the form

$$\frac{1}{e_1} \frac{1}{e_2} \frac{1}{e_3} \dots \frac{1}{e_L} \quad (23)$$

where e_i is the amended propagator from the definition of $V_c(0)$. Each one is a random variable that depends on the values of the $\{E_i\}$. Let $P(\Pi)d\Pi$ denote the probability that the product takes value Π .

It is well known that for a lattice with coordination number Z , the number of non-repeating paths of length L leading from any given atom scales like K^L where K is of the same order as Z . Then we get the result

$$n(T)dT = K^L P(T/V^L) dT/V^L = (K/V)^L P(T/V^L) dT \quad (24)$$

More generally, [2] finds

$$n(T)dT = [F(K, W/V)]^L L(T) dT/T^2 \quad (25)$$

The probability of the sum is then²

$$P(\Sigma)d\Sigma \sim F^L(K, W/V) L(\Sigma) d\Sigma/\Sigma \quad (26)$$

Localization is when the most probable value of Σ is of order one, or $F^L(K, (W/V)_0) L(\Sigma) = 1$, for appropriate $(W/V)_0$. We see that this series will converge so long as $(W/V) > (W/V)_0$, which is the desired result.

Let us summarize what we've found: There is a critical value for the ratio between the disorder potential to the bandwidth, above which **all** solutions to Schrodinger's equation are (unlike Bloch states) localized³. Evaluating this critical disorder strength is difficult, though later numerical and experimental estimates suggest that a value of two is typical for s states (in three dimensions), around one for d states⁴, and about four thirds for two dimensional systems [3].

For the nearest neighbor or tight-binding model, we obtain that if the dimension of the lattice $d \leq 2$, then the states are localized for arbitrary disorder strength. This result is reinforced through

²Here we are using that we approximate the sum by its largest terms.

³Though electrons can still be mobile via thermally activated hopping.[3]

⁴Reasonable since d states share fewer neighbors than s states.

the scaling theory of localization, developed later by Anderson and others. For $d \geq 3$, we have that the states are localized provided that the disorder strength exceeds its critical value [2].

Another remarkable consequence of this theorem is that it gives a concrete example of a thermodynamically large system that fails to come to equilibrium. The model itself has no external bath and fails to act as one for any sub-system. The introduction of an external bath would likely lead to transport in this model. The result reiterates that we cannot just assume thermal equilibrium in any large system.

3 Further theoretical work

Mott showed that some states can still remain localized even below this critical disorder strength [4]. Even in low disorder regimes, there exists an energy ε_c below which the states are localized and above which the states are extended. This energy came to be referred to later as “the mobility edge”. If the Fermi energy ε_f exceeds the mobility edge, then the system is metallic. On the other hand, if $\varepsilon_f < \varepsilon_c$, the system is insulating, and the conductivity σ is zero [3].

For concreteness, the way the Fermi level is controlled experimentally is through the addition of donors in the doped semiconductor. As we add impurities, the density of free carriers increases, and with it the Fermi energy. Thus, a precondition to Anderson’s theorem is that we are at sufficiently low carrier density. At high enough densities, the Fermi level may exceed the mobility edge, and so extended states will appear.

The nature of the transition between the insulating and conducting phases of the material as we vary $\varepsilon \equiv \varepsilon_f - \varepsilon_c$ was highly contentious. Whether the transition is first or second order was a source of fierce debate between Anderson, who (with others) predicted that the conductivity vanished continuously, and Mott, who suggested that it was discontinuous. The argument given by the infamous gang of four, of which Anderson was a member, settled the argument for most of the physics community. Using scaling theory, it found that in $d = 2$ dimensions, there is no true metallic conductivity [5], agreeing with earlier experiments and numerical simulations. Indeed, this result is even robust to adding weak interactions between the electrons [6].

The essence of the argument goes as follows: we know from elementary classical electrodynamics that the dimensionless conductance $g = \frac{\hbar}{e^2} G \equiv \frac{\hbar}{e^2 R}$ (where R is the resistance) scales like $g(L) \sim \sigma L^{d-2}$ for conducting materials, where L is the characteristic length scale. For small g , localization is valid and the conductance drops off exponentially $g(L) \sim e^{-L/\xi}$. We can thus compute the flow under the RG scheme⁵ and find that

$$\beta(g) \equiv \frac{L dg}{g dL} = \begin{cases} D - 2 + \dots & g \gg 1 \\ \log g + \dots & g \ll 1 \end{cases} \quad (27)$$

The resulting flow diagram, found in [5], shows no critical points for $d = 2$ and flows always to $g = 0$, the insulating phase.

A natural generalization of the phenomena of disorder induced localization is what happens when we introduce interactions to the system. When localization persists with the addition of interactions between the mobile entities, we have *many-body* localization, and it is an active area of

⁵The RG step consists of gluing together hypercubes of the material at their boundaries.

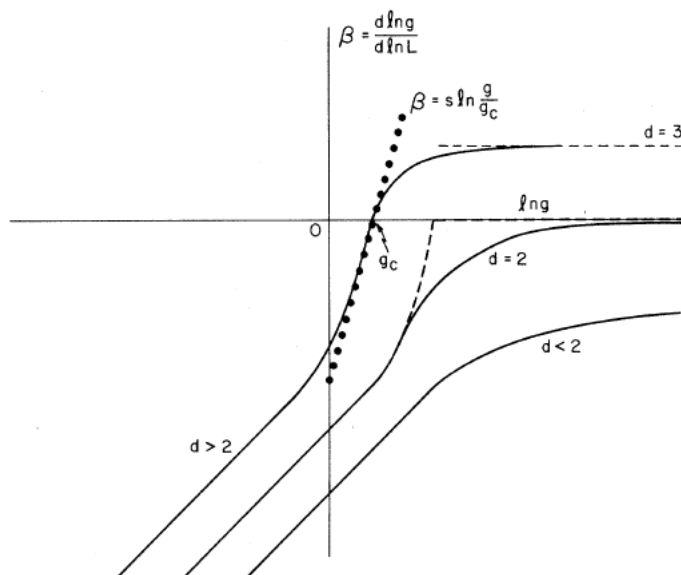


Figure 1: The flow diagram for the scaling theory of localization provided in [5]. Notice the absence of a critical point in $d = 2$ dimensions.

research even today. Many-body localization (MBL) is interesting not only because of its connection to Anderson localized systems, but also because many-body localized systems defy the eigenstate thermalization hypothesis, which is often used to justify statistical mechanics from quantum mechanics. Thus, it is of extreme relevance for the foundations of quantum statistical mechanics. MBL is one half of a phase diagram, the other being the thermalized phase, where the order parameter, the entanglement entropy, goes continuously from area law to volume law as we vary the disorder strength. Furthermore, many-body localization is promising for realizing finite temperature quantum memory [7]. A rigorous proof of a metal-insulator transition in a weakly-interacting many-body system at finite temperature was done in [8].

4 Experimental confirmations of localization

Electron localization was the concern of Anderson's original argument, though it seemed for decades to be particularly difficult to verify. By 1982, the variation of the conductivity σ with carrier density n was reliably measured for uncompensated samples. It was found that the transition was very sharp and but that σ follows a power law in $(n - n_c)/n_c$ as predicted by scaling theory, though with a different exponent. Earlier work also found that samples with larger compensation seemed to agree with scaling theory more, suggesting that Coulomb interaction effects between the carriers are non-negligible [9].

In 1982, [10] studied low temperature phosphorous-doped silicon semiconductors in three dimensions near the metal-insulator transition at the critical carrier density. They were able to confirm that the transition is indeed continuous and highly sensitive to $n - n_c$, obeying power law. The detailed results can be glimpsed at in fig. 2.

In 1988, Pruisken established that further experimental confirmation can be found in, unexpect-

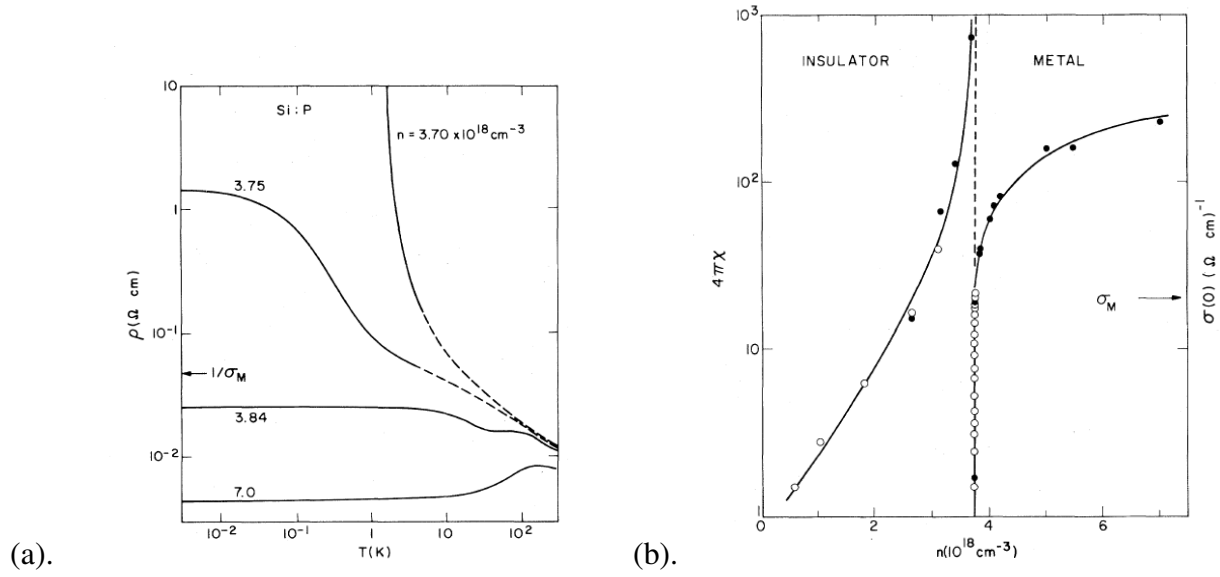


Figure 2: (a). Here, resistivity ρ is plotted as a function of temperature for several values of the carrier density n near the critical value. The bottom most curve (with the greatest density n) is clearly metallic, as the resistivity decreases with decreasing T and approaches a small finite value as T goes to zero. The next one above is a insulator-like, with conductivity above Mott's supposed minimum conductivity $\sigma_M \approx 20[\Omega \cdot \text{cm}]^{-1}$. The one above that one is very near the transition and has a conductivity *below* Mott's theoretical minimum. It is also insulator-like, though we refrain from calling it an insulator because it has non-zero conductivity when extrapolated to $T = 0$. The upper-most curve is evidently at the insulating side of the transition. Note the sensitivity to carrier density. (b) We will focus on the right hand side, which presents the $T = 0$ conductivity in the metallic phase. The solid line is the conductivity suggested by scaling theory at $T = 0$ with $\sigma(0) = \sigma_c((n/n_c) - 1)^{\nu}$ and $\nu = 0.55 \pm 0.1$. Note the samples with conductivity less than σ_M . The findings also hold for non-zero T [10].

edly, the integer quantum hall effect (QHE). In the QHE, the conductivity increases like a staircase as a function of the magnetic field. The jumps in conductivity occur when the Fermi level of the conducting electrons approaches the Landau level. Using scaling theory, Pruisken shows that the electrons are localized at energies away from the Landau levels and extended near them [11].

A plethora of examples of localization have been discovered in recent years using classical waves. Unlike electrons, photons are non-interacting and also easier to control. By manufacturing materials that strongly scatter light, for example certain powders that high refractive indices and tiny grain sizes, experimentalists have been able to disrupt the normal diffusion of light and localize waves. The goal is to minimize the mean free path of the photons without absorbing them, for example by using light whose frequency is smaller than the bandgap of the semi-conducting powder but that still has a high refractive index. This was done in [12], the results of which can be seen in fig. 3 below.

Since Anderson's 1958 paper, localization has remained an active area of research with many remaining open problems. The nature of the many-body localization transition still remains a mystery, and no analytic derivations of the critical exponents yet exist. Localization is continuing

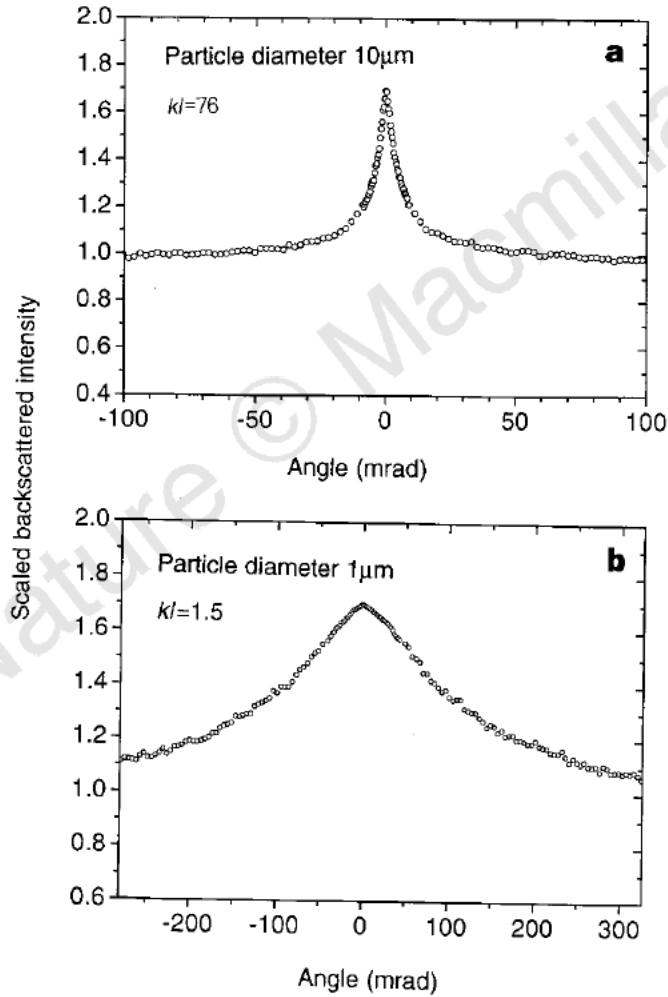


Figure 3: Coherent backscattering from the GaAs powder sample. The sharp peak in intensity shows the localization of light in the medium. The sample was illuminated with “a mode-locked Nd:YAG laser operating at 76 MHz, with a wavelength of 1,064 nm, pulse duration 100 ps, beam diameter 6mm and incident power 100mW” [12].

to be discovered in ubiquitous theoretical models and disparate experimental contexts. It still challenges our understanding of the foundations of quantum statistical mechanics and the approach to thermodynamic equilibrium. It may hold the key to realizing quantum computers. Anderson laid the groundwork for the rich world of localization physics, and, in a way, brought order from disorder.

References

- [1] G. Feher and E. A. Gere, “Polarization of phosphorus nuclei in silicon,” *Phys. Rev.*, vol. 103, pp. 501–503, Jul 1956. [Link](#).
- [2] P. W. Anderson, “Absence of diffusion in certain random lattices,” *Phys. Rev.*, vol. 109, pp. 1492–1505, Mar 1958. [Link](#).
- [3] S. N. Mott, F. R. S., M. Pepper, S. Pollitt, R. H. Wallis, and C. J. Adkins, “The anderson transition,” *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 345, no. 1641, pp. 169–205, 1975. [Link](#).
- [4] N. Mott, “Electrons in disordered structures,” *Advances in Physics*, vol. 16, no. 61, pp. 49–144, 1967. [Link](#).
- [5] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, “Scaling theory of localization: Absence of quantum diffusion in two dimensions,” *Phys. Rev. Lett.*, vol. 42, pp. 673–676, Mar 1979. [Link](#).
- [6] B. L. Altshuler, A. G. Aronov, and P. A. Lee, “Interaction effects in disordered fermi systems in two dimensions,” *Phys. Rev. Lett.*, vol. 44, pp. 1288–1291, May 1980. [Link](#).
- [7] R. Nandkishore and D. A. Huse, “Many-body localization and thermalization in quantum statistical mechanics,” *Annual Review of Condensed Matter Physics*, vol. 6, no. 1, pp. 15–38, 2015. [Link](#).
- [8] D. Basko, I. Aleiner, and B. Altshuler, “Metalinsulator transition in a weakly interacting many-electron system with localized single-particle states,” *Annals of Physics*, vol. 321, no. 5, pp. 1126 – 1205, 2006. [Link](#).
- [9] G. A. Thomas, Y. Ootuka, S. Katsumoto, S. Kobayashi, and W. Sasaki, “Evidence for localization effects in compensated semiconductors,” *Phys. Rev. B*, vol. 25, pp. 4288–4290, Mar 1982. [Link](#).
- [10] T. F. Rosenbaum, R. F. Milligan, M. A. Paalanen, G. A. Thomas, R. N. Bhatt, and W. Lin, “Metal-insulator transition in a doped semiconductor,” *Phys. Rev. B*, vol. 27, pp. 7509–7523, Jun 1983. [Link](#).
- [11] A. M. M. Pruisken, “Universal singularities in the integral quantum hall effect,” *Phys. Rev. Lett.*, vol. 61, pp. 1297–1300, Sep 1988. [Link](#).
- [12] D. S. Wiersma, P. Bartolini, A. Lagendijk, and R. Righini, “Localization of light in a disordered medium,” *Nature*, vol. 390, pp. 671 EP –, Dec 1997. [Link](#).