Notions and Results Regarding Locality in Quantum Systems

Ahmed Akhtar

March 19, 2019

Contents

1 Introduction and Motivation

Discrete quantum systems are indispensable in the study of the solids. Such many-body systems are described by a set of sites $1...N$, each with a local Hilbert space $\mathbb{H}_i$ of finite dimension D, and a Hamiltonian H that acts on the full space $\mathbb{H} = \otimes_{i=1}^{N} \mathbb{H}_i$ of dimension D^N. Often, we are interested in calculating ground state correlation functions of some observables O_i, such as $\langle \psi_0 | O_i O_j | \psi_0 \rangle$, where $|\psi_0\rangle$ is the lowest energy state of H. At low temperatures, these correlation functions agree with their thermal values.

A common feature of quantum many-body systems is *locality*. This means the Hamiltonian is a sum of terms, where the operator norm of each term decays with the diameter of the set it is supported on. The rate at which it decays could be exponential, power law, or, more commonly, k-local[1]. In relativity and field theory, locality is often manifest, so that the causal influence of some

[1] This means that only sites within distance k are coupled together in the Hamiltonian. We will define terms more precisely in the next section.

space-time region A is limited to its light-cone. Does this feature *emerge* in quantum many-body systems whose Hamiltonians are local in the sense we have described? It is not so clear, as we know from perturbation theory that a k-local, $k > 1$, interaction might couple sites arbitrarily far away in an arbitrarily short amount of time. For example, consider two operators O_X, A_Y, supported on far away spacetime regions X, Y, respectively. From the Baker-Campbell-Hausdorff formula and elementary quantum mechanics, the time evolution of an operator O_X is

$$O_X(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} [H, ...[H, O_X]...] \tag{1}$$

where the commutator with H is taken n times. Then, as long as there is a sequence of terms in the Hamiltonian connecting the two regions, for $t \neq 0$, it is possible for $[O_X(t), A_Y] \neq 0$, even if $ct << \text{dist}(X,Y)$ [1].

It turns out, however, that a "light-cone" does emerge, outside of which correlation functions decay rapidly. The inequalities satisfied by the correlation functions are called Leib-Robinson bounds, and velocity of "light" the Leib-Robinson velocity. Furthermore, one can show a remarkable number of other properties of local Hamiltonians using these bounds, such as the protection of topological order and a form of Goldstone's theorem. One such application is in proving the famed Leib-Schultz-Mattis theorem in higher dimensions. We will follow the treatment in [2].

2 A Leib-Robinson bound and proof

2.1 Some Basic Definitions

To talk about locality precisely, we need to introduce a metric for our system. If we can label the sites of our full Hilbert space $1...N$, let $\text{dist}(i, j)$ denote the distance between those sites. Then $\text{dist}(i, j)$ satisfies the usual properties of a metric on a metric space, namely that it is non-negative, zero only when $i = j$, and satisfies the triangle inequality. For a 1d lattice, commonly referred to as a chain, we could have in mind $\text{dist}(i, j) = |i - j|$, and in the case of periodic boundary conditions, $\text{dist}(i, j) = min_n|i - j + nN|$.

Let $\Lambda = \{1, ...N\}$ refer to all the sites in our system, and $A, B \subset \Lambda$ to subsets of our system. We define the **distance** between these regions as

$$\text{dist}(A,B) = \min_{i \in A, j \in B} \text{dist}(i, j) \tag{2}$$

and the **diameter** of a region A as

$$\text{diam}(A) = \max_{i,j \in A} \text{dist}(i, j) \tag{3}$$

We say an operator O is **supported** on $A \subset \Lambda$ if

$$O = I_{\Lambda \setminus A} \otimes P_A \tag{4}$$

where $I_{\Lambda \setminus A}$ is the identity on sites not in A, and P_A acts on the sites in A. We may sometimes say O is "only supported" on A to mean that A is the minimal set on which O is supported.

Now, suppose we can write the Hamiltonian as

$$H = \sum_Z H_Z \tag{5}$$

where $Z \subset \Lambda$, and H_Z is supported on Z. Then, H is **finite-ranged** or **k-local** if

$$\mathrm{diam}(Z) > k \implies ||H_Z|| = 0 \tag{6}$$

where $||H_Z||$ is the operator norm[2].

2.2 Statement and proof

We will prove a bound on correlation functions between operators $A_X(t), B_Y$. We will assume that H is time-independent so the system has time-translation invariance and we don't need to time evolve B_Y. Furthermore, we will assume that the operators H_Z that make up H decay exponentially with their diameter. We are working in the Heisenberg picture so that

$$A_X(t) := e^{iHt} A_X e^{-iHt} \tag{7}$$

Lastly, we consider the bosonic case, so we are bounding the commutator. However, the fermionic case is identical if we switch the commutator with the anti-commutator. The following theorem was first rigorously proved in [3].

Theorem 1. *Suppose A_X, B_Y are supported on regions X,Y, respectively, with* $\mathrm{dist}(X,Y) > 0$. *Then*

$$||[A_X(t), B_Y]|| \leq 2||A_X||||B_Y|||X|e^{-\mu\,\mathrm{dist}(X,Y)}\left(e^{2s|t|} - 1\right) \tag{8}$$

for positive constants μ, s satisfying

$$\sum_{X \ni i} ||H_X|||X|e^{\mu\,\mathrm{diam}(X)} \leq s < \infty \qquad \forall i \in \Lambda \tag{9}$$

Proof. First note that this theorem applies for all systems defined on finite graphs with finite-ranged or exponentially decaying interactions. The thermodynamic limit can be performed by taking N to infinity, but ought to be done carefully.

First, we can discretize the time evolution in steps of $\varepsilon = t/N'$, and $t_n = \varepsilon n$.

$$||[A(t), B]|| - ||[A(0), B]|| = \sum_{n=0}^{N'-1} \varepsilon \frac{||[A(t_{n+1}), B]|| - ||[A(t_n), B]||}{\varepsilon} \tag{10}$$

Now we want to bound the numerator in the above expression. First note that we can rewrite the first term in the numerator as $[A(t_{n+1}), B] = [e^{iHt_n} A(\varepsilon) e^{-iHt_n}, B] = e^{iHt_n}[A(\varepsilon), B(-t_n)]e^{-iHt_n}$. By a similar argument, $[A(t_n), B] = e^{iHt_n}[A, B(-t_n)]e^{-iHt_n}$. Furthermore, multiplication by a unitary cannot change an operator's norm. Using these two facts and a triangle inequality, we can expand numerator in powers of ε. First, we'll define I_X as the terms in the Hamiltonian whose support intersects with X i.e.

[2]For example, for H_Z hermitian, $||H_Z|| = \max_{\psi, |\psi|=1} |H_Z \psi|$.

$$I_X = \sum_{Z:Z\cap X\neq\phi} H_Z \tag{11}$$

Then, following the algebra in [2], we can bound the change in the norm of the commutator over time. We can then take $\varepsilon \to 0$, $N' \to \infty$, and take the sum to an integral. Of course, this is only valid if $H_Z(t)$ is a continuous function of t. This is easy to verify for most cases.

$$\begin{aligned}
||[A(t_{n+1}),B]|| - ||[A(t_n),B]|| &= ||[A(\varepsilon),B(-t_n)]|| - ||[A,B(-t_n)]|| && (12)\\
&= ||[A + i\varepsilon[H,A],B(-t_n)]|| - ||[A,B(-t_n)]|| + \mathscr{O}(\varepsilon^2) && (13)\\
&= ||[A + i\varepsilon[I_X,A],B(-t_n)]|| - ||[A,B(-t_n)]|| + \mathscr{O}(\varepsilon^2) && (14)\\
&= ||[e^{i\varepsilon I_X} A e^{-i\varepsilon I_X},B(-t_n)]|| - ||[A,B(-t_n)]|| + \mathscr{O}(\varepsilon^2) && (15)\\
&= ||[A, e^{-i\varepsilon I_X} B(-t_n) e^{i\varepsilon I_X}]|| - ||[A,B(-t_n)]|| + \mathscr{O}(\varepsilon^2) && (16)\\
&\leq \varepsilon||[A,[I_X,B(-t_n)]]|| + \mathscr{O}(\varepsilon^2) && (17)\\
&\leq 2\varepsilon||A||||[I_X,B(-t_n)]|| + \mathscr{O}(\varepsilon^2) && (18)
\end{aligned}$$

$$\begin{aligned}
||[A(t),B]|| - ||[A(0),B]|| &\leq 2||A|| \sum_{n=0}^{N'-1} \varepsilon||[I_X,B(-t_n)]|| + \mathscr{O}(\varepsilon^2) && (19)\\
&\leq \sum_{Z:Z\cap X\neq\phi} \sum_{n=0}^{N-1} \varepsilon||[H_Z(t_n),B]|| + \mathscr{O}(\varepsilon^2) && (20)
\end{aligned}$$

$$\boxed{||[A(t),B]|| - ||[A,B]|| \leq 2||A|| \sum_{Z:Z\cap X\neq\phi} \int_0^{|t|} ds||[H_Z(s),B]||} \tag{21}$$

The remainder of the proof is somewhat tedious and not particularly instructive, so we will sketch it in broadly instead. Interested readers can read the argument in [4]. The rest of the proof goes as such: first, notice that if we divide the Leib-Robinson bound by $||A_X||$, the RHS is totally independent of both A_X. This means we can take the supremum of the quantity on the LHS divided by $||A_X||$.

$$C_B(X,t) := \sup_{A\in\mathbb{A}_X} \frac{||[A(t),B]||}{||A||} \tag{22}$$

where $\mathbb{A}_X$ is the set of observables supported on X. The boxed bound above becomes

$$C_B(X,t) \leq C_B(X,0) + 2 \sum_{Z_1:Z_1\cap X\neq\phi} ||H_{Z_1}|| \int_0^{|t|} ds_1 C_B(Z_1,s_1) \tag{23}$$

The quantity $C_B(X,t)$ is bounded above by $2||B||$, and is 0 at $t = 0$ if $X \cap Y = \phi$. Since $\text{dist}(X,Y) > 0$, we can therefore drop the first term in the above equation. We can then apply the bound iteratively to $C_B(Z,t)$.

$$C_B(X,t) \leq 2 \sum_{Z_1 : Z_1 \cap X \neq \phi} ||H_{Z_1}|| \int_0^{|t|} ds_1 C_B(Z_1, 0)$$

$$+ 2^2 \sum_{\substack{Z_1 \cap X \neq \phi \\ Z_2 \cap Z_1 \neq \phi}} ||H_{Z_1}|| ||H_{Z_2}|| \int_0^{|t|} ds_1 \int_0^{|s_1|} ds_2 C_B(Z_2, s_2) \quad (24)$$

$$C_B(X,t) \leq 2||B|| \left(2|t| \sum_{\substack{Z_1 \cap X \neq \phi \\ Z_1 \cap Y \neq \phi}} ||H_{Z_1}|| + \frac{(2|t|)^2}{2!} \sum_{Z_1 \cap X \neq \phi} ||H_{Z_1}|| \sum_{\substack{Z_2 \cap Z_1 \neq \phi \\ Z_2 \cap Y \neq \phi}} ||H_{Z_2}|| + \ldots \right) \quad (25)$$

The first term is bounded by $2||B||(2|t|)\sum_{i \in X} \exp(-\mu \operatorname{dist}(i,Y)) \leq 2||B||(2|t|)|X| \exp(-\mu \operatorname{dist}(X,Y))$. The higher order terms can be similarly bounded. $\square$

3 Understanding the bounds physically

3.1 Emergent causal structure

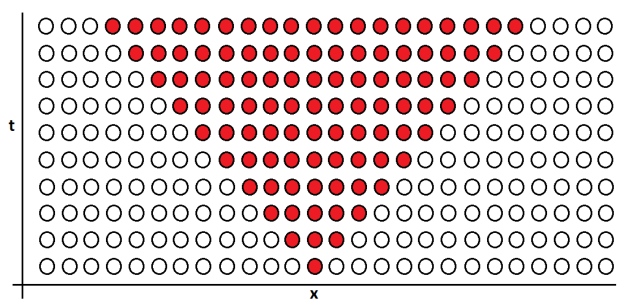

Figure 1: The "light-cone" for information propogation through a one dimensional lattice. The slope is given by the Leib-Robinson velocity. An operator initially supported on X can be well-approximated by an operator supported on the red horizontal cross-section a time t later.

The Leib-Robinson bound proved in the last section allows us to introduce the notion of a light-cone in systems with exponentially decaying interactions (see 1). Any operator A_X initially supported on $X \subset \Lambda$ can be arbitrarily well approximated a time t later by an operator $A_X^l(t)$ supported on sites within a distance $l = v_{LR}t$ of X. The choice of v_{LR} is somewhat arbitrary, one is $v_{LR} = 4s/\mu$.

To see this, first let $B_l(X) = \{i \in \Lambda : \operatorname{dist}(i,X) \leq l\}$ i.e. the set of sites within a distance l of X. Although $A_X(t)$ may have support outside the light cone, we can integrate out its support outside of it, so that we have an operator $A_X^l(t)$ supported only on $B_l(X)$.

$$A_X^l(t) = \int dU U A_X(t) U^\dagger \quad (26)$$

where we are integrating over unitaries supported on $\Lambda \setminus B_l(X)$ with the Haar measure. This is only supported on $B_l(X)$ since, for any unitary G supported outside of $B_l(X)$

$$GA_X^l(t)G^\dagger = \int dU GUA_X(t)U^\dagger G^\dagger = \int d(GU)(GU)A_X(t)(GU)^\dagger = A_X^l(t) \tag{27}$$

Then, since $UA_X(t)U^\dagger = A_X(t) + U[A_X(t), U^\dagger]$ and (after normalizing the Haar measure) applying the Leib-Robinson bound shows $A_X^l(t)$ rapidly converges to $A_X(t)$ within the cone.

$$||A_X^l(t) - A_X(t)|| = ||\int dU U[A_X(t), U^\dagger]|| \leq \int dU ||[A_X(t), U]|| \leq 2||A_X|||X|e^{-\mu l}\left(e^{2s|t|} - 1\right) \tag{28}$$

$$= 2||A_X|||X|e^{-\mu(l - 2s|t|/\mu)}\left(1 - e^{-2s|t|}\right) \leq 2||A_X|||X|e^{-2\mu(l - v_{LR}|t|/2)} \tag{29}$$

We can even rewrite the bound in terms of the Leib-Robinson velocity. Let $l = \text{dist}(X,Y), v_{LR} = 4s/\mu$, and suppose $t \leq l/v_{LR}$, then we have $2st \leq \mu l/2$, and $e^{-\mu l}(e^{2st} - 1) \leq e^{-\mu l/2} \leq e^{-\mu l/2}\frac{v_{LR}|t|}{l}$ so that

$$||[A_X(t), B_Y]|| \leq \frac{v_{LR}|t|}{l} g(l)|X|||A_X||||B_Y|| \tag{30}$$

where $g(l)$ is a function that decays exponentially in l. This is similar to a result found in Lorentz-invariant field theories which shows that if the theory is local, then the field-correlator vanishes for spacelike displacements, and causality is manifest. The fact that a similar result emerges in generic quantum many-body systems is fascinating, and even has connections to formulating quantum gravity in terms of quantum many-body systems [5, 6, 7].

The bound can be made even more tight in the case of finite-range interactions. Let us consider the transverse field Ising model on a d dimensional cubic lattice. Let X and Y be single-sites separated by a distance l.

$$H = -J \sum_{<ij>} S_i^z S_j^z + B \sum_i S_i^x \tag{31}$$

The bound on $C_B(X,t)$ becomes a sum over paths starting at X and ending at Y.

$$||[A(t), B]|| \leq 2||A||||B||\frac{(2|t|)^l}{l!}\frac{l!}{\prod_{i=1}^d (X_i - Y_i)!}|J/4|^l + \mathscr{O}\left(J^{d+1}\right) \tag{32}$$

Using techniques from statistical mechanics, one finds in this case that $g(l)$ decays faster than exponentially ($\sim e^{-al^2}$) [2].

3.2 Bounds on correlation functions

Another straightforward application of the Leib-Robinson bounds is to ground state correlation functions i.e. things we can measure experimentally. The theorem and proof can be found in [8]. It says for a quantum lattice with a unique ground state and spectral gap ΔE

$$|\langle\psi_0, A_X B_Y \psi_0\rangle - \langle\psi_0, A_X \psi_0\rangle\langle\psi_0, B_Y \psi_0\rangle| \leq C||A_X||||B_Y||\left(\exp(-l\Delta E/2v_{LR}) + \min(|X|,|Y|)g(l)\right) \tag{33}$$

where A_X,B_Y are any operators supported on $X,Y \subset \Lambda$, $l = \text{dist}(X,Y)$, and C is some constant. If there is a ground state sector with exponentially small splitting and a gap above the ground-state sector

$$|\langle\psi_0^a, A_X B_Y \psi_0^a\rangle - \langle\psi_0^a, A_X P_0 B_Y \psi_0^a\rangle| \leq C||A_X||||B_Y||\left(\exp(-l\Delta E/2v_{LR}) + \min(|X|,|Y|)g(l)\right) \tag{34}$$

where $P_0 = \sum_a |\psi_0^a\rangle\langle\psi_0^a|$ is the projector onto the ground-state sector. The proof can be found in [8], [2], but will be left out for brevity. An application of the theorem to the transverse field Ising model can be obtained in the limit $B << J$, when there are two nearly-degenerate ground states $|\psi_0^{\pm}\rangle$ (indexed by their eigenvalue $\prod_{i=1}^N 2S_i^x$) and a gap.

$$|\psi_0^{\pm}\rangle = \frac{1}{2}\left(|... \uparrow\uparrow\uparrow ...\rangle \pm |... \downarrow\downarrow\downarrow ...\rangle\right) \tag{35}$$

The matrix representation M of S_i^z in the ground state sector can be written in terms of the order parameter m as

$$M = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \tag{36}$$

Here, m is the magnetization per site. For $B = 0$, $m = 1/2$, and for $B > 0$, $m > 0$ in the ordered phase. Then, $\langle\psi_0^+, S_i^z P_0 S_j^z \psi_0^+\rangle = m^2$, and so $\langle\psi_0^+, S_i^z S_j^z \psi_0^+\rangle$ is exponentially close to m^2 as $\text{dist}(i,j) \to \infty$.

4 The Leib-Schultz-Mattis Theorem in arbitrary dimensions

The bounds derived in the previous sections can be applied to derive a type of Goldstone's theorem [9], and also to show the stability of topological order [10]. For brevity's sake, we will restrict ourself to just one other remarkable consequence of locality: the Leib-Schultz-Mattis theorem. The theorem is about quantum systems of spins with certain symmetries and it implies they are gapless. In class, we talked mainly about systems with gapless excitations (e.g. plasmons, phonons, etc.). Whether a system is gapless or not has significant physical ramifications, determining, for example, if it's a metal or an insulator.

To prove the Leib-Schultz-Mattis (LSM) theorem, we must first define the quasi-adiabatic continuation operator. We will use the continuation operator to traverse paths in paramater space. The key idea here is that continuing along gapped paths keeps us in the ground state.

4.1 Quasi-Adiabatic continuation

Suppose we have a parameter dependent *local* Hamiltonian[3] $H_s = \sum_Z H_Z(s)$. We will assume $H_Z(s)$ is differentiable. The transverse field Ising model is an example, with the parameter $s =$

[3]For example, it may be exponentially local, so that the bound we derived is applicable.

B/J. If the system has a unique ground state $\psi_0(s)$ and a non-zero gap $\Delta E(s) > \Delta E > 0 \forall s$, then we can define a local operator, called the quasi-adiabatic continuation operator, such that $iD_s\psi_0(s) = \partial_s\psi_0(s)$ (or $iD_s\psi_0(s) \approx \partial_s\psi_0(s)$, depending on what we want to prove).

Given H_s, an operator O, and a function $F(t)$, define the **quasi-adiabatic continuation operator** by

$$iD(H_s, O) := \int F(\Delta E t)\exp(iH_s t)O\exp(-iH_s t)dt \tag{37}$$

F will satisfy different properties depending on whether we want the exact continuation operator or the approximate one. To ensure D is hermitian, F is an odd function of time. Then, of course, its Fourier transform $\tilde{F}(\omega) = 0$ at $\omega = 0$. Given a parameter dependent Hamiltonian $H_s = \sum_Z H_Z(s)$, define

$$D_s := D(H_s, \partial_s H_s) = \sum_Z D_s^Z \tag{38}$$

The exact adiabatic continuation operator D_s, which is the one we will need for LMS theorem, has that $\tilde{F}(\omega) = -1/\omega$ for $|\omega| \geq 1$. Now we can check that D_s satisfies the required property. In the second line, we insert the identity. In the third line, we use the eigenstate property of $\psi_0(s)$. In the fourth line, we use that $\tilde{F}(\omega) = -1/\omega$ for $|\omega| \geq 1$. The last equality is a well-known result from ordinary perturbation theory.

$$\begin{aligned}
iD_s\psi_0(s) &= \int F(\Delta E t)\exp(iH_s t)(\partial_s H_s)\exp(-iH_s t)dt\,\psi_0(s) &(39)\\
&= \sum_{i=0} |\psi_i(s)\rangle\langle\psi_i(s)| \int F(\Delta E t)\exp(iH_s t)(\partial_s H_s)\exp(-iH_s t)dt\,\psi_0(s) &(40)\\
&= \sum_{i=0} |\psi_i(s)\rangle\langle\psi_i(s), (\partial_s H_s)\psi_0(s)\rangle \int F(\Delta E t)\exp(i(E_s - E_0)t)dt &(41)\\
&= \sum_{i\neq 0} \frac{|\psi_i(s)\rangle\langle\psi_i(s), (\partial_s H_s)\psi_0(s)\rangle}{E_0(s) - E_i(s)} = \partial_s\psi_0(s) &(42)
\end{aligned}$$

Equivalently, if $P_0(s)$ is the projector onto the ground state sector (potentially degenerate),

$$\partial_s P_0(s) = i[D_s, P_0(s)] \tag{43}$$

Using the Leib-Robinson bounds, we can determine the locality of D_s. This fact is essential to the proof of the LSM theorem. To prove this, we will require that F decays superpolynomially[4], and that $||H_Z(s)||, ||\partial_s H_Z(s)||$ decay superpolynomially in $\mathrm{diam}(Z)$. This is obviously satisfied by all finite-ranged Hamiltonians. To prove the locality of D_s, first consider each term D_s^Z.

$$D_s^Z = \int F(\Delta E t)(\partial_s H_Z(s))(t)dt \tag{44}$$

As we showed earlier, $(\partial_s H_Z(s))(t)$ can be supported by an operator O_l supported on $B_l(Y)$ where Y is the support of $\partial_s H_Z$ and $l \sim v_{LR}t$. This is the reason why D_s^Z is local. More formally,

[4] You might be wondering if such a function $F(t)$ exists! It does, in fact, though we will leave its existence proof to those more interested in the mathematical details.

$$D_s^Z = \int F(\Delta E t) O_\infty dt = \sum_{l=0}^{\infty} \int F(\Delta E t)(O_l - O_{l-1}) dt \tag{45}$$

Now, evidently, D_s^Z is a sum of terms $D_Y(s)$ that decay superpolynomially in Y. Thus, the full operator can be written as a sum of terms whose norm decays superpolynomially in the size of their support.

$$D_s = \sum_Y D_Y(s) \qquad ||D_Y(s)|| \in \mathcal{O}\big(\mathrm{diam}(Y)^{-n}\big) \,\forall n \tag{46}$$

4.2 Proof in arbitrary dimensions

We are now ready to prove the Leib-Schultz-Mattis theorem. It was first proven in [11], though the more recognizable form of it, considering a system with a $U(1)$ symmetry, appears in [12]. The full proof of the theorem is involved, and what follows is only intended to be a sketch of the proof, but the main ideas are present nonetheless. The higher dimensional proof for a system with $SU(2)$ symmetry is given in [8].

The theorem was first proved in 1961 in the case of a one-dimensional periodic chain [11]. It bounded the gap at const. $/L$. Extending the proof to higher dimensions has been difficult, namely because the system can be short-ranged or long-ranged [13]. Essentially, in higher dimensions, there exist two distinct ways of creating low-energy excitations: in the case of short-range correlations, we can dimerize the system into resonating valence bond states, and then construct a low-energy state similar to the "twisted" ψ_{LSM} in [14]; conversely, in the case of long-range correlations, we have low energy spin-wave excitations. This contrasts the one-dimensional case, where there is no long range order.

First, we need to define the (hopefully familiar) notion of a conserved charge Q. A Hamiltonian H has a conserved charge Q if $Q = \sum_{i\in\Lambda} q_i$, where q_i is supported on site i, has integer eigenvalues, and is bounded $||q_i|| < q_{max}$, and $[H,Q] = 0$.

Theorem 2. *Let $H = \sum_Z H_Z$ be an R-local Hamiltonian with a conserved charge Q defined on a finite dimensional lattice with periodic boundary conditions (PBC) and translation invariance (TI) in at least one direction and let L be the lattice size in that direction. Further, suppose each term is bounded by J, i.e. $||H_Z|| < J$, and that the total number of sites N is bounded by a constant times a polynomial in L. Define the ground state filling factor ρ by*

$$\rho = \langle \psi_0, Q\psi_0 \rangle / L \tag{47}$$

Then $\rho \notin \mathbb{Z}$ implies either the ground state is degenerate or the gap between the ground state and first excited state is bounded by

$$\Delta E \leq \mathrm{const.} \log(L)/L \tag{48}$$

where the constant only depends on R, J, q_{max}, and the lattice structure.

Proof. Here's the idea: we use the existence of a gap ΔE to construct a low energy state orthogonal to the ground state. If the gap exceeds $\log(L)/L$, then the variational state will have energy lower than $\log(L)/L$ above the ground state, thus proving (by contradiction) that either gap is bounded or the ground state is degenerate. The existence of the gap and locality suggest the system's insensitivity to boundary conditions, and so we form this state by "twisting" the boundary conditions at $x = 0$ and $x = L/2$. This state is referred to as *topologically excited*, since, as we will see, it has the same expectation values for all local operators as the ground state.

We begin by defining some quantities and operators that will be useful later in the proof. Let $x(i)$ denote the coordinate of site $i \in \Lambda$ along the direction with PBC and TI. For example, the vertical line at x would be denoted as $VL_x = \{i : x(i) = x\} \subset \Lambda$. Let Q_X denote the charge in one half of the lattice i.e.

$$Q_X = \sum_{i:1\leq x(i)\leq L/2} q_i \tag{49}$$

Furthermore, define a parameter dependent family of Hamiltonians

$$H(\theta_1,\theta_2) = \sum_Z H_Z(\theta_1,\theta_2) \tag{50}$$

where

$$H_Z(\theta_1,\theta_2) = \begin{cases} e^{i\theta_1 Q_X} H_Z e^{-i\theta_1 Q_X} & \text{dist}(Z, VL_0) \leq R \\ e^{-i\theta_2 Q_X} H_Z e^{i\theta_2 Q_X} & \text{dist}(Z, VL_{L/2}) \leq R \\ H_Z & \text{dist}(Z, VL_0 \cup VL_{L/2}) > R \end{cases} \tag{51}$$

Since the charge takes integer values, $H(0,0) = H(2\pi,-2\pi) = H$. Note also that H and $H(\theta,-\theta)$ share the same spectrum because they're related by basis change:

$$H(\theta,-\theta) = e^{i\theta Q_X} H e^{-i\theta Q_X} \tag{52}$$

This is clear because for H_Z that are supported between vertical lines but R away, $H_Z = e^{i\theta Q} H_Z e^{-i\theta Q} = e^{i\theta Q_X} H_Z e^{-i\theta Q_X}$, since the charges outside the strip commute with H_Z, and if H_Z is instead supported outside the strip a distance R from the vertical lines, $e^{-i\theta Q} H_Z e^{i\theta Q} = e^{i\theta Q_X} H_Z e^{-i\theta Q_X}$.

We will now use the adiabatic continuation operator to continue along three paths in parameter space. These paths are actually closed loops when we think of the parameter space as being a torus, and the first two are the generators of the fundamental group of the torus. Note that since all paths start at $(0,0)$, all of the initial Hamiltonians are gapped, but this does not necessarily mean they remain gapped! Only for path three will the Hamiltonian unequivocally remain gapped, because the spectrum is invariant along this path.

1. θ_1 evolves from 0 to 2π and $\theta_2 = 0$. We are evolving $H_\theta = H(\theta,0)$, the path is generated by D^1_θ, and the unitary relating the initial and final states is $W_1 = \exp\left(\int_0^{2\pi} d\theta D^1_\theta\right)$ (with W_1 θ-ordered).

2. θ_2 evolves from 0 to 2π and $\theta_1 = 0$. We are evolving $H_\theta = H(0,-\theta)$, the path is generated by D^2_θ, and the unitary relating the initial and final states is $W_2 = \exp\left(\int_0^{2\pi} d\theta D^2_\theta\right)$ (with W_2 θ-ordered).

3. $\theta_1 = -\theta_2 = \theta$ evolves from 0 to 2π. We are evolving $H_\theta = H(\theta,-\theta)$, the path is generated by D_θ, and the unitary relating the initial and final states is $W = \exp\left(\int_0^{2\pi} d\theta D_\theta\right)$ (with W θ-ordered).

If the continuation operators are local, and the gap is sufficiently big, then $W_1 W_2$, $W_2 W_1$, and W are all close in their operator norm.

$$W_1 W_2 \approx W_2 W_1 \approx W \tag{53}$$

The idea is this: W_1 only affects sites near VL_0, and is weak outside of a width inversely proportional to the energy gap ΔE. The inverse of the gap sets a length scale for decay, and thus if we want to approximate W_1 up to a distance $L/4$ of the line, $1/\Delta E$ must be small compared to L. We find that the gap needs to be at least $f(l)/L$, where $f(l)$ grows more slowly than any polynomial, in order for the above approximation to be valid.

Finally, we are ready to construct our variational state with energy lower than the supposed gap.

$$\psi_1 = W_1 \psi_0 \tag{54}$$

The energy of this state is close to E_0, because either H_Z is far from VL_0, in which case

$$\langle W_1 \psi_0, H_Z W_1 \psi_0 \rangle \approx \langle \psi_0, H_Z \psi_0 \rangle \tag{55}$$

or it is close to VL_0 and so it is far from $VL_{L/2}$, and so H_Z approximately commutes with W_2

$$\langle W_1 \psi_0, H_Z W_1 \psi_0 \rangle \approx \langle W_2 W_1 \psi_0, H_Z W_2 W_1 \psi_0 \rangle = \langle W \psi_0, H_Z W \psi_0 \rangle = \langle \psi_0, H_Z \psi_0 \rangle \tag{56}$$

Since we did not assume the form of H_Z, we have also shown here why the excited state is referred to as topological–they are indistinguishable by local operators. Now, we have left to show that ψ_1 is orthogonal to the ground state. We will show this by considering its eigenvalue under the translation operator T. Suppose $\langle \psi_0, T\psi_0 \rangle = z$. Then

$$\begin{aligned}
\langle \psi_0, W_1^\dagger T W_1 \psi_0 \rangle &= z \langle \psi_0, W_1^\dagger (T W_1 T^\dagger) \psi_0 \rangle && (57)\\
&\approx z \langle \psi_0, W_2^\dagger W_1^\dagger (T W_1 T^\dagger) W_2 \psi_0 \rangle && (58)\\
&= z w^* \langle \psi_0, (T W_1 T^\dagger) W_2 \psi_0 \rangle && (59)\\
&\approx z w^* w' && (60)\\
&= z \langle \psi_0, \exp\left(2\pi i \sum_{i:x(i)=1} q_i \right) \psi_0 \rangle && (61)\\
&= z \langle \psi_0, \exp(2\pi i Q/L) \psi_0 \rangle && (62)\\
&= z \exp(2\pi i \rho) \neq z && (63)
\end{aligned}$$

- In the first equality, we insert the identity $T^\dagger T$.
- In the second equality, we use that W_2 approximately commutes with $T W_1 T^\dagger$, which is the same as W_1 but shifted over one.
- In the third equality, we let w be the eigenvalue for $W \approx W_1 W_2$ (we know ψ_0 is an eigenstate because $\langle \psi_0, W^\dagger H W \psi_0 \rangle = \langle \psi_0, H \psi_0 \rangle$).
- In the fourth equality, we let w' refer to the eigenvalue for $(T W_1 T^\dagger) W_2$.

- In the fifth equality, we use the explicit values of w and w'.
- In the sixth line, we use translation invariance to rewrite the exponential in terms of the charge Q.
- In the seventh, we plug in the definition of ρ and we use that its not an integer.

Let us summarize what we have done. We assumed the initial system, $H = H(0,0)$, had a gap. Then, assuming the gap was big enough at $\theta = 0$, we constructed the low-energy topologically excited state which is orthogonal to the ground state. This is similar to the degenerate ground state sector in the Majumdar-Ghosh model. So, either the ground state is unique and the gap is bounded by const. $\log L/L$, or it is degenerate. $\square$

5 Concluding remarks and new horizons

The study of lattice models with local Hamiltonians is indispensable to the study of solids, and much of what we've studied here can be applied to a wide range of models (e.g. tight-binding model, Heisenberg model, transverse field Ising model) describing solids. Statements about the existence (or lack thereof) of gaps in the spectrum of a Hamiltonian, such as LSM theorem, can tell you whether the corresponding system is an conductor or an insulator.

The Leib-Robinson bounds reveal much of the physics of quantum lattices. They can be used to uncover a sort of causal structure in these lattices. This causal structure can be used to prove a number of remarkable theorems regarding quantum lattices. One such theorem is the Leib-Schultz-Mattis theorem, which was originally proven in 1d. The bounds provide a proof that works for all finite dimensional lattices–a major achievement! The bounds also show that the size of the gap sets a length scale over which correlations decay exponentially. The bigger the gap, the faster the ground state connected correlation function of two operators will decay. In part due to the generality of the arguments in [2], the machinery we developed can be put towards understanding generic quantum many-body systems, so long as they are local in some sense. For these reasons and others, the Leib-Robinson bounds continue to be relevant in the frontiers of research and may even provide insight into new discoveries.

The commutators of local hermitian operators, such as in theorem 1, are effective for characterizing the "butterfly effect" in quantum many-body systems [1]. A quantum many-body system is said to be chaotic if the commutator of any two, spatially seperated local operators grows large and remains so after the "scrambling time." For example, for large N gauge theories with N^2 degrees of freedom at each site, we find for short times

$$-\langle [W(x,t), V(0)]^2 \rangle_\beta = \frac{K}{N^2} e^{\lambda_L (t - x/v_B)} + \mathscr{O}\left(N^{-4}\right) \tag{64}$$

where v_B is the butterfly velocity.

The relationship between v_{LR} and v_B is an active area of research. The form of the bound in theorem 1 is in terms of microscopic quantities and hence it is UV-sensitive and *state-independent*. A tighter bound for the low-energy physics might be obtained by looking at the matrix elements of the commutator between the low-lying states, such as in the above equation. We can therefore view the butterfly velocity as the low-energy Leib-Robinson velocity, but its usefulness hardly stops there. Even high-energy scattering in the vicinity of black holes has been shown to be connected to quantum chaos [15].

The notion of the light-cone in quantum-many body lattice systems has also inspired work in quantum gravity. Much effort has been made in tensor network formulations of space-time to understand how insertions of local operators into a tensor network representing some d dimenion-sional space-time affects probes some time later. By studying commutators, one can show that these structures also exhibit an emergent causal structure, similar to the one studied in this report [5, 7].

There is also work on the quantum information side of things. The stability of topologically ordered phases to perturbations in the Hamiltonian is tied to the prospect of quantum computers. Suppose we start off with a Hamiltonian H_0 with a topologically non-trivial phase separated by a gap (e.g. the toric code, Levin-Wen model, etc.) and we add a perturbation V whose strength is parameterized by s

$$H = H_0 + sV \tag{65}$$

Does the model remain in the topologically non-trivial phase as we vary s? Does the gap remain open? For $||V||$, L finite, it can be easily shown that the gap remains open up to some ball around $s = 0$ [2]. Recent work using the quasi-adiabatic continuation operators [16, 17] provide bounds which are uniform functions of L.

However, we still know relatively little about topological entanglement entropy in the presence of local perturbations. Topological entanglement entropy is a constant in the entanglement entropy that depends only on the topology of the space on which the Hamiltonian acts. Understanding the behavior of entanglement entropy in various lattice models might help to clarify the nature and stability of topological phases.

The growth of entanglement entropy is also useful in understanding the many-body localization (MBL) transition. Much recent work has been done in understanding the growth of out-of-time-order-correlators (OTOCs) [18, 19] to determine how and if systems thermalize. Similarly, the entanglement entropy of many-body systems can be used as an order parameter in the MBL-ETH transition [20]. Perhaps using ideas or techniques highlighted in this report could bring us closer to understanding many-body localization through a fuller understanding of how the entanglement entropy evolves in lattice models.

References

[1] D. A. Roberts and B. Swingle, "Lieb-robinson and the butterfly effect," 2016.

[2] M. B. Hastings, "Locality in quantum systems," 2010.

[3] E. H. Lieb and D. W. Robinson, "The finite group velocity of quantum spin systems," *Comm. Math. Phys.*, vol. 28, no. 3, pp. 251–257, 1972.

[4] M. B. Hastings and T. Koma, "Spectral gap and exponential decay of correlations," 2005.

[5] J. Cotler, X. Han, X.-L. Qi, and Z. Yang, "Quantum causal influence," 2018.

[6] P. Hayden, S. Nezami, X.-L. Qi, N. Thomas, M. Walter, and Z. Yang, "Holographic duality from random tensor networks," 2016.

[7] X.-L. Qi and Z. Yang, "Space-time random tensor networks and holographic duality," 2018.

[8] M. B. Hastings, "Lieb-schultz-mattis in higher dimensions," *Phys. Rev. B*, vol. 69, p. 104431, Mar 2004.

[9] M. B. Hastings and X.-G. Wen, "Quasi-adiabatic continuation of quantum states: The stability of topological ground state degeneracy and emergent gauge invariance," 2005.

[10] S. Bravyi, M. B. Hastings, and F. Verstraete, "Lieb-robinson bounds and the generation of correlations and topological quantum order," 2006.

[11] E. Lieb, T. Schultz, and D. Mattis, "Two soluble models of an antiferromagnetic chain," *Annals of Physics*, vol. 16, no. 3, pp. 407 – 466, 1961.

[12] I. Affleck and E. H. Lieb, "A proof of part of haldane's conjecture on spin chains," *Letters in Mathematical Physics*, vol. 12, pp. 57–69, Jul 1986.

[13] G. Misguich, C. Lhuillier, M. Mambrini, and P. Sindzingre, "Degeneracy of the ground-state of antiferromagnetic spin-1/2 hamiltonians," 2001.

[14] N. E. Bonesteel, "Valence bonds and the lieb-schultz-mattis theorem," *Phys. Rev. B*, vol. 40, pp. 8954–8960, Nov 1989.

[15] S. H. Shenker and D. Stanford, "Black holes and the butterfly effect," 2013.

[16] S. Bravyi, M. Hastings, and S. Michalakis, "Topological quantum order: stability under local perturbations," 2010.

[17] S. Bravyi and M. B. Hastings, "A short proof of stability of topological order under local perturbations," 2010.

[18] R. Fan, P. Zhang, H. Shen, and H. Zhai, "Out-of-time-order correlation for many-body localization," 2016.

[19] "New horizons towards thermalization," *Nature Physics*, vol. 14, no. 10, pp. 969–969, 2018.

[20] A. Lukin, M. Rispoli, R. Schittko, M. E. Tai, A. M. Kaufman, S. Choi, V. Khemani, J. Lonard, and M. Greiner, "Probing entanglement in a many-body-localized system," 2018.